OTS: 60-41096

JPRS: 52

3 August 1960

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# TOWARDS THE QUESTION CONCERNING THE EXPANSION BY A SMALL PARAMETER OF THE SOLUTION OF THE SYSTEM OF EQUATIONS OF HYDRO-THERMODYNAMICS APPLICABLE TO ATMOSPHERIC PROCESSES

by G. I. Marchu and N. M. Kireeva

-USSR-

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JPRS: 5203
CSO: 4603-N

# TOWARDS THE QUESTION CONCERNING THE EXPANSION BY A SMALL PARAMETER OF THE SOLUTION OF THE SYSTEM OF EQUATIONS OF HYDRO-THERMODYNAMICS APPLICABLE TO ATMOSPHERIC PROCESSES

[This is a translation of an article by G.I. Marchuk and N.M. Kireeva, in Trudy, Inst. Fiz. Atmos., 1958 (work done in 1953); CSO: 4603-N]

As the solutions of the non-linear system of hydrodynamic equations applicable to atmospheric processes presents exceptional difficulties of a mathematic character, usually they are limited to the search for time derivatives of pressure, temperature and the vertical velocity of air particles.

As is well known in the quasigeostrophic approximation the mentione functions enter into the system of differential equations linearly and could be calculated for highly general assumptions relative to the structure of thermobaric fields by methods, at present, greatly exploited [1]. However the possibilities of using the quasi-geostrophic approximation are restricted and frequently quite insufficient for describing the evolution of the fields of meteorological elements.

In the present work, the authors set as an aim the consistent carrying out of I. A. Kibel's idea concerning the possibility of the expansion of the solution of the general problem of hydro-thermodynamics applicable to atmospheric processes through degrees of a small parameter, taking into account three dimensions and the vertical baroclinicity of the atmosphere.

The paragraphs of articles 1 - 3 were written by G. I. Marchuk, paragraph 4 by N. M. Kireeva; she also carried out the calculations of

the influence functions $M_5$ and $M_6$ .

## § I Formulation of the Problem

If the $x$ axis of a Cartesian coordinate system is directed ALONG THE LATITUDE CIRCLES TOWARD THE EAST; THE $y$ AXIS IS along the meridians toward the north, $\zeta = p/P$ is vertically upward, where $p$ is the pressure at any height, while $P$ is the pressure at earth's surface, then the system of hydro-thermodynamic equations applicable to atmospheric processes takes the following form:

$$\frac{du}{dt} + \frac{\tau}{P}\frac{\partial u}{\partial \zeta} - lv = -g\frac{\partial z}{\partial x} \tag{1.1}$$

$$\frac{dv}{dt} + \frac{\tau}{P}\frac{\partial v}{\partial \zeta} + lu = -g\frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{P}\frac{\partial \tau}{\partial \zeta} = 0$$

$$\frac{dT}{dt} - \frac{\gamma_a - \gamma}{P g} RT\frac{\tau}{\zeta} = 0$$

$$T = -\frac{g}{R}\,\zeta\,\partial z/\partial \zeta$$

where $u$ and $v$ are the components of the vector velocity of air particles, along the $x$ and $y$ axes

$\tau$ is the vertical velocity relative to isobaric surfaces

$\zeta =$ constant

$z$ is the height of isobaric surfaces above sea level

$T$ is temperature

$g$ is the force of gravity

$l$ is $2\,\omega \sin\phi$

$\omega$ is the angular velocity of the earth's rotation

$\phi$ is geographic latitude

$R$ is the gas constant

$\gamma$ is the vertical gradient of temperature,

$$\gamma_a = \frac{A_g}{c_p}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Let us take into consideration non-dimensional quantities, which we shall designate by the index 1.

Let $x = Lx_1, \quad y = Ly_1, \quad t = \frac{L}{U}t_1, \quad u = Uu_1,$

$$\tau = \frac{U^2 P}{\ell L^2}\tau_1, \quad z = \frac{\ell L U^2}{U g} z_1, \quad T = \frac{\gamma_a - \gamma}{g} R\bar{T} \cdot T_1$$

where $\bar{T}$ is the mean temperature of the atmosphere. Then the hydrodynamic equations take the form

$$\varepsilon \frac{du_1}{dt_1} + \varepsilon^2 \tau_1 \frac{du_1}{d\zeta} - v_1 = -\frac{\partial z_1}{\partial x_1}$$

(1.2)
$$\varepsilon \frac{dv_1}{dt_1} + \varepsilon^2 \tau_1 \frac{\partial v_1}{\partial \zeta} + u_1 = -\frac{\partial z_1}{\partial y_1}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} + \varepsilon \frac{\partial \tau_1}{\partial \zeta} = 0$$

$$\frac{dT_1}{dt_1} - \varepsilon \frac{\tau_1}{\zeta} = 0, \quad T_1 = -\frac{1}{m_1^2}\varepsilon \zeta \frac{\partial z_1}{\partial \zeta},$$

where $\varepsilon = \frac{U}{L\ell}$ is a non-dimensional parameter, while $m_1^2 = \frac{1}{L^2}\frac{\gamma_a - \gamma}{g\ell^2}R^2\bar{T}$. For standard atmospheric conditions and for mean latitudes, the coefficient $m_1^2$ is roughly equal to 0.5.

If we take $L = 10^6\, m$, $U = 10\, m/sec$, $\gamma_a - \gamma = 0.3 \times 10^{-2}\, deg/m$, $\ell = 10^{-4}$, then we obtain the following characteristic values for variations of meteorological elements

$$\delta\tau = \frac{U^2 P}{\ell L^2} = 10\, mb/day$$

which corresponds to a vertical velocity of $1 \; cm/sec$ ,

$$\delta z = \frac{lL}{U} \cdot \frac{U^2}{g} = 10^2 m$$

$$\delta T = \frac{\gamma_a - \gamma}{g} RT = 20 deg$$

From this estimate it follows, that the considered characteristic scales of the phenomena lead to reasonable order of magnitude values of the meteorological elements; for this $\delta t = \frac{L}{U} = 10^5 \; sec.$ , while the non-dimensional parameter $\mathcal{E} = U/lL = 10^{-1}$.

From the first two equations of (1.2), by cross differentiation we obtain the equation, which connects the individual derivative of vorticity with the velocity divergence:

$$(1.3) \qquad \mathcal{E} \frac{d}{dt}\Omega + (1 + \mathcal{E}\Omega) D + \mathcal{E}^2 \left( \tau \frac{\partial \Omega}{\partial \zeta} + \frac{\partial \tau}{\partial x}\frac{\partial v}{\partial \zeta} - \frac{\partial \tau}{\partial y}\frac{\partial u}{\partial \zeta} \right) = 0$$

where

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Here and everywhere in the future, for the sake of simplicity we shall drop the index 1 for the non-dimensional quantities.

In ~~our~~ AN analogous way, differentiating the first equation of system (1.2) with respect to $x$ , and the second with respect to $y$ and then adding them, we obtain a new equation:

$$(1.4) \qquad \mathcal{E} \frac{dD}{dt} - \Omega + \mathcal{E}\,\Omega' + \mathcal{E}^2 \left( \tau \frac{\partial D}{\partial \zeta} + \frac{\partial \tau}{\partial x}\frac{\partial u}{\partial \zeta} + \frac{\partial \tau}{\partial y}\frac{\partial u}{\partial \zeta} \right) = -\Delta z$$

where
$$\Omega' = \frac{\partial}{\partial x}\left(\frac{du}{dt}\right) + \frac{\partial}{\partial y}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Using the third equation of the system (1.2), one can rewrite (.13) and (1.4) in the following form:

(1.5) $$\frac{d\Omega}{dt} - \frac{\partial \tau}{\partial \zeta} = \varepsilon\left(\tau\frac{\partial \Omega}{\partial \zeta} - \Omega\frac{\partial \tau}{\partial \zeta} + \frac{\partial \tau}{\partial x}\frac{\partial v}{\partial \zeta} - \frac{\partial \tau}{\partial y}\frac{\partial u}{\partial \zeta}\right)$$

(1.6) $$\Omega - \Delta z = -\varepsilon\,\Omega' - \varepsilon^2\left(\frac{\partial \tau}{\partial \zeta} + \frac{\partial \tau}{\partial x}\frac{\partial u}{\partial \zeta} + \frac{\partial \tau}{\partial y}\frac{\partial u}{\partial \zeta}\right) - \varepsilon^2\tau\frac{\partial \tau}{\partial \zeta}$$

Putting (1.6) into (1.5) and neglecting terms, the order of smallness which are higher than $\varepsilon$, we obtain

(1.7) $$\frac{\partial \Delta z}{\partial t} + u\frac{\partial \Delta z}{\partial x} + v\frac{\partial \Delta z}{\partial y} - \frac{\partial \tau}{\partial \zeta} = -\varepsilon\left(\frac{d\Omega'}{dt} + \Delta z\frac{\partial \tau}{\partial \zeta} - \tau\frac{\partial \Delta z}{\partial \zeta} + \frac{\partial \tau}{\partial x}\frac{\partial v}{\partial \zeta} - \frac{\partial \tau}{\partial y}\frac{\partial u}{\partial \zeta}\right) + O(\varepsilon^2)$$

where $O(\varepsilon)$ is a symbol, which indicates the order of magnitude of the neglected terms.

Let us recall, that

(1.8) $$u = -\frac{\partial z}{\partial y} + \varepsilon u' \qquad v = \frac{\partial z}{\partial x} + \varepsilon v'$$

where

(1.9) $$u' = -\frac{dv}{dt} = -\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) - \varepsilon\frac{dv'}{dt}$$
$$v' = \frac{du}{dt} = -\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) + \varepsilon\frac{du'}{dt}$$

It is possible to write the system of equations (1.9) in complex form

(1.9') $\varepsilon \, dw/dt - i w = dF/dt$ where $w = u' + v'$

$F = \partial \zeta/\partial y - i \, \partial \zeta/\partial x$

The solution of this equation would permit the recovery of the field of vector velocity through the field of the function $\zeta$, however the exact integration of equation (1.9') is not presented in the present article as a subject of investigation. For our aims, it is sufficient to state, that from (1.9) with an accuracy to small quantities of first order it follows, that

$$u' = \bar{u}' + \varepsilon u''$$
$$v' = \bar{v}' + \varepsilon v''$$

where

$$\bar{u}' = -\left[\frac{\partial}{\partial x}\left(\frac{\partial \zeta}{\partial t}\right) + (\zeta, \zeta_x)\right]$$

$$\bar{v}' = -\left[\frac{\partial}{\partial y}\left(\frac{\partial \zeta}{\partial t}\right) + (\zeta, \zeta_y)\right]$$

$$u'' = -u'\frac{\partial^2 \zeta}{\partial x^2} - v'\frac{\partial^2 \zeta}{\partial x \partial y} - \frac{dv'}{dt}$$

$$v'' = -u'\frac{\partial^2 \zeta}{\partial x \partial y} - v'\frac{\partial^2 \zeta}{\partial y^2} + \frac{du'}{dt}$$

and the symbol

$$(\phi, \psi) = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x}$$

Therefore, relation (1.8) with an accuracy to small quantities of second order can be rewritten in the form

$$\left.\begin{aligned} u &= -\frac{\partial \zeta}{\partial y} + \varepsilon \bar{u}' + O(\varepsilon^2) \\ v &= \frac{\partial \zeta}{\partial x} + \varepsilon \bar{v}' + O(\varepsilon^2) \end{aligned}\right\} \quad (1.8')$$

Having (1.8') in this form, (1.7) can be brought to the following form:

(1.10)
$$\frac{\partial \Delta z}{\partial t} + \frac{\partial \tau}{\partial \zeta} = -(z, \Delta z) - \varepsilon\left[\frac{\partial \overline{\Omega}'}{\partial t} + (z, \overline{\Omega}') + \right.$$
$$\left. + \overline{u}'\frac{\partial \Delta z}{\partial x} + \overline{v}'\frac{\partial \Delta z}{\partial y} + \Delta z\frac{\partial \tau}{\partial \zeta} - \tau\frac{\partial \Delta z}{\partial \zeta} + \frac{\partial \tau}{\partial x}\frac{\partial^2 z}{\partial x \partial \zeta} + \frac{\partial \tau}{\partial y}\frac{\partial^2 z}{\partial y \partial \zeta}\right] + O(\varepsilon^2),$$

where
$$\overline{\Omega}' = \frac{\partial \overline{v}'}{\partial x} - \frac{\partial \overline{u}'}{\partial y} = -2\left(z_{xx} z_{yy} - z_{xy}^2\right)$$

Let us now consider the equation of heat flux

(1.11)
$$\frac{dT}{dt} - \varepsilon\frac{\tau}{\zeta} = 0$$

which, considering the hydrostatic equation with an accuracy to small values of second order, could be presented in the following form:

(1.12)
$$\frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial t}\right) + m_1^2\frac{\tau}{\zeta^2} = -\left(z, \frac{\partial z}{\partial \zeta}\right) - \varepsilon\left[\overline{u}'\frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial x}\right) + \overline{v}'\frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial y}\right)\right] + O(\varepsilon^2)$$

Thus, for the determination of the functions $\partial z/\partial t$ and $\tau$ we obtain a system of two partial differential equations:

(1.13)
$$\frac{\partial \Delta z}{\partial t} + \frac{\partial \tau}{\partial \zeta} = B + \varepsilon B' + O(\varepsilon^2)$$

(1.14)
$$\frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial t}\right) + m_1^2\frac{\tau}{\zeta^2} = A + \varepsilon A' + O(\varepsilon^2)$$

where
$$B = -(z, \Delta z), \quad A = -\left(z, \frac{\partial z}{\partial \zeta}\right)$$

$$B' = -\left[\frac{\partial \Omega'}{\partial t} + (z, \Omega') + \pi' \frac{\partial \Delta z}{\partial x} + \nu' \frac{\partial \Delta z}{\partial y} - \right.$$
(1.15)
$$\left. - \Delta z \frac{\partial z}{\partial \zeta} + \tau \frac{\partial \Delta z}{\partial \zeta} + \frac{\partial \tau}{\partial x}\frac{\partial^2 z}{\partial x \partial \zeta} + \frac{\partial \tau}{\partial y}\frac{\partial^2 z}{\partial y \partial \zeta}\right],$$

$$A' = -\left[\pi' \frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial x}\right) + \nu' \frac{\partial}{\partial \zeta}\left(\frac{\partial z}{\partial y}\right)\right].$$

Finally, eliminating the function $\tau$ from equations (1.13) and (1.14), we obtain a second order differential equation for the calculation of the function $\frac{\partial z}{\partial t}$ [1]

(1.16)
$$\left(\frac{\partial}{\partial \zeta} \zeta^2 \frac{\partial}{\partial \zeta} - m_1^2 \Delta\right)\frac{\partial z}{\partial t} = m_1^2 B + \frac{\partial}{\partial \zeta} \zeta^2 A + \\ + \varepsilon\left(m_1^2 B' + \frac{\partial}{\partial \zeta}\zeta^2 A'\right) + O(\varepsilon$$

---

1 I. A. Kibel's "first approximation" will follow from (1.16) for th assumption, that the quantity $m_1^2$ is the order $\varepsilon$, i.e. for conditions, when $\gamma_a \approx \gamma$.

---

It is immediately clear, that the principal and small first order terms in equation (1.16) have been delineated in an obvious form. For the integration of equation (1.16) it is still necessary to give the boundary conditions, which we shall take in the following form:

(1.17)
$$\tau = 0 \quad \text{at } \zeta = 0, \quad \tau = g\rho \frac{\partial z}{\partial t} \quad \text{at } \zeta = 1.$$

Using the equation of heat flux, it is possible to show, that due to condition (1.17) and a quite natural assumption concerning the character of the thermobaric fields in the lower troposphere, we obtain the following boundary condition for $\partial z/\partial t$ [1]:

$$\left(\varsigma \frac{\partial}{\partial \varsigma} + \alpha\right)\frac{\partial z}{\partial t} = A + \varepsilon A' + O(\varepsilon^2) \quad at \quad \varsigma = 1 \tag{1.18}$$

$$\lim_{\varsigma \to 0} \varsigma^2 \frac{\partial}{\partial \varsigma}\left(\frac{\partial z}{\partial t}\right) = 0$$

where

$$\alpha = \frac{R(\gamma_a - \gamma)}{g} \tag{1.19}$$

In agreement with this, the solution of equation (1.16) is presented in the following form

$$\left(\frac{\partial z}{\partial t}\right)_0 = \iiint_{(D)} B M_1(\varsigma, \eta, r')\,dD + \frac{1}{m_1^2}\iiint_{(D)} A M_2(\varsigma, \eta, r')\,dD \tag{1.20}$$

$$\left(\frac{\partial z}{\partial t}\right)_0 = \iiint_{(D)} B' M_1(\varsigma, \eta, r')\,dD + \frac{1}{m_1^2}\iiint_{(D)} A' M_2(\varsigma, \eta, r')\,dD$$

where $M_1(\varsigma, \eta, r')$ and $M_2(\varsigma, \eta, r')$ are influence functions [1],

$$r' = \sqrt{(x-x')^2 + (y-y')^2},$$

$dD$ is an element of volume in the Cartesian system of coordinates $(x, y, \varsigma)$.

Thus, the solution of the problem is obtained by means of formula (1.20) with an accuracy to quantities of second order of smallness.

It is immediately clear, that all the unknown quantities, which

are contained in $A'$ and $B'$, could be expressed through $\left(\partial z/\partial t\right)_0$ and $\tau$, obtained from the solution in the first approximation.

However, due to the fact, that the functions $\frac{d\overline{\Omega}}{d\tau}$, $\overline{u'}$, $\overline{v'}$ for the use of $\left(\frac{\partial z}{\partial t}\right)_0$ are found from differences, to a considerable degree, of mutually exclusive quantities, it is expedient to compute the quantities with the help of more stable formulas in regard to the computation.

M. I. Iudin, using the quasi-geostrophic approximation of the vector velocity of air particles for the calculation of $\overline{u'}$ and $\overline{v'}$ formulated a second order equation, which he integrated approximately with the help of the полупрямых method.

In the following paragraph, we shall obtain the general expression for $\overline{u'}$ and $\overline{v'}$ in the form of space integrals of the functions of the fields of meteorological elements with the corresponding influence functions.

## II. Calculation of the functions $\overline{u'}$ and $\overline{v'}$

Let us look for the functions $\overline{u'}$ and $\overline{v'}$ in the following form:

$$\overline{u'} = -\frac{\partial\psi}{\partial y} + \iiint\limits_{(D)} U(x', y', \eta)\, G(\xi, \eta, r')\, dD$$

$$\overline{v'} = \frac{\partial\psi}{\partial x} + \iiint\limits_{(D)} V(x', y', \eta)\, G(\xi, \eta, r')\, dD \tag{2.1}$$

where $\psi$, $U$, $V$ are unknown functions of the fields of meteorological elements,

$G$ is the influence function, the form of which also must

be determined.

Then, let us differentiate the first relation in (2.1) with respect to $x$ and the second with respect to $y$, then we obtain:

$$
(2.2)\qquad
\begin{aligned}
\frac{\partial \overline{u'}}{\partial x} &= -\frac{\partial^2 \psi}{\partial x \partial y} + \iiint_{(D)} U \frac{\partial G}{\partial x} \, dD \\
\frac{\partial \overline{v'}}{\partial y} &= \frac{\partial^2 \psi}{\partial x \partial y} + \iiint_{(D)} V \frac{\partial G}{\partial y} \, dD
\end{aligned}
$$

If we assume, that $G \longrightarrow 0$ where $r \longrightarrow \infty$, then let us integrate by parts and we obtain:

$$
(2.3)\qquad
\begin{aligned}
\iiint_{(D)} U \frac{\partial G}{\partial x} \, dD &= \iiint_{(D)} \frac{\partial U}{\partial x'} G \, dD \\
\iiint_{(D)} V \frac{\partial G}{\partial y} \, dD &= \iiint_{(D)} \frac{\partial V}{\partial y'} G \, dD
\end{aligned}
$$

Therefore,

$$
(2.4)\qquad
\begin{aligned}
\frac{\partial \overline{u'}}{\partial x} &= -\frac{\partial^2 \psi}{\partial x \partial y} + \iiint_{(D)} \frac{\partial U}{\partial x'} G \, dD \\
\frac{\partial \overline{v'}}{\partial y} &= \frac{\partial^2 \psi}{\partial x \partial y} + \iiint_{(D)} \frac{\partial V}{\partial y'} G \, dD
\end{aligned}
$$

Adding the obtained two equations (2.4), we have:

$$
(2.5)\qquad \frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{v'}}{\partial y} = \iiint_{(D)} \left( \frac{\partial U}{\partial x'} + \frac{\partial V}{\partial y'} \right) G \, dD
$$

But, on the other hand

$$
(2.6)\qquad \frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{v'}}{\partial y} = -\frac{\partial \tilde{\tau}}{\partial \zeta}
$$

Therefore

(2.7) $$\frac{\partial \tilde{\tau}}{\partial \rho} = -\iiint_{(D)} \left( \frac{\partial U}{\partial x'} + \frac{\partial V}{\partial y'} \right) G \, dD$$

For the calculation of $\partial \tilde{\tau} / \partial \rho$, let us take advantage of the expression for $\tilde{\tau}$, which in non-dimensional quantities will have the following form:

(2.8) $$\tilde{\tau} = \frac{m_1^2}{\varepsilon} \iiint_{(D)} \left[ \Delta(T, z) + (T, \Delta z) + (z, \Delta T) \right] \rho M_3 (\rho, \eta, r) \, dD$$

where $\rho M_3$ is an influence function [1].

Let us differentiate (2.8) with respect to $\rho$ :

(2.9) $$\frac{\partial \tilde{\tau}}{\partial \rho} = \frac{m_1^2}{\varepsilon} \iiint_{(D)} \left[ \Delta(T, z) + (T, \Delta z) + (z, \Delta T) \right] \frac{\partial}{\partial \rho} (\rho M_3) \, dD$$

Let us note, that

(2.10) $$\Delta (T, z) + (T, \Delta z) + (z, \Delta T) = 2 \left[ \frac{\partial}{\partial x} (T, z_x) + \frac{\partial}{\partial y} (T, z_y) \right]$$

Let us substitute (2.10) into (2.9):

(2.11) $$\frac{\partial \tilde{\tau}}{\partial \rho} = \frac{2 m_1^2}{\varepsilon} \iiint_{(D)} \left[ \frac{\partial}{\partial x'} (T, z_{x'}) + \frac{\partial}{\partial y'} (T, z_{y'}) \right] M_5 \, dD$$

where

$$M_5 = \frac{\partial}{\partial \rho} (\rho M_3)$$

Next, putting (2.11) into (2.7), we obtain

$$\frac{2m_1^2}{\varepsilon}\iiint_{(D)}\left[\frac{\partial}{\partial x'}(T,\mathfrak{z}_{x'})+\frac{\partial}{\partial y'}(T,\mathfrak{z}_{y'})\right]M_5\,dD=-\iiint_{(D)}\left[\frac{\partial U}{\partial x'}+\frac{\partial V}{\partial y'}\right]G\,d \tag{2.12}$$

From (2.12) it follows, that the equation is satisfied, if we put

$$\left.\begin{aligned} G &= M_5 \\ U &= -\frac{2m_1^2}{\varepsilon}\left[(T,\mathfrak{z}_x)+\frac{\partial\phi}{\partial y}\right] \\ V &= -\frac{2m_1^2}{\varepsilon}\left[(T,\mathfrak{z}_y)-\frac{\partial\phi}{\partial x}\right] \end{aligned}\right\} \tag{2.13}$$

where $\phi$ is a new unknown function.

Thus, with the calculation of the expressions (2.13) $\overline{u'}$ and $\overline{V'}$ are written down this way:

$$\begin{aligned} \overline{u'} &= -\frac{\partial\psi}{\partial y}-\frac{2m_1^2}{\varepsilon}\iiint_{(D)}\left[(T,\mathfrak{z}_{x'})+\frac{\partial\phi}{\partial y'}\right]M_5\,dD \\ \overline{V'} &= \frac{\partial\psi}{\partial x}-\frac{2m_1^2}{\varepsilon}\iiint_{(D)}\left[(T,\mathfrak{z}_{y'})-\frac{\partial\phi}{\partial x'}\right]M_5\,dD \end{aligned} \tag{2.14}$$

It remains to determine the unknown functions $\phi$ and $\psi$.
For this purpose, let us differentiate the second equation of (2.14) with respect to $x$ and the first with respect to $y$. Then after integration by parts we obtain:

$$\begin{aligned} \frac{\partial\overline{V'}}{\partial x} &= \frac{\partial^2\psi}{\partial x^2}-\frac{2m_1^2}{\varepsilon}\iiint_{(D)}\left[\frac{\partial}{\partial x'}(T,\mathfrak{z}_{y'})-\frac{\partial^2\phi}{\partial x'^2}\right]M_5\,dD \\ \frac{\partial\overline{u'}}{\partial y} &= -\frac{\partial^2\psi}{\partial y^2}-\frac{2m_1^2}{\varepsilon}\iiint_{(D)}\left[\frac{\partial}{\partial y'}(T,\mathfrak{z}_{x'})+\frac{\partial^2\phi}{\partial y'^2}\right]M_5\,dD \end{aligned} \tag{2.15}$$

Subtracting поЧленно one of the equations from the other, we obtain:

(2.16) $$\iiint_{(D)} \left[ \frac{\partial}{\partial x'} (T_1 \zeta_{y'}) - \frac{\partial}{\partial y'} (T_1 \zeta_{x'}) - \Delta \phi \right] M_5 \, dD = 0$$

Here at the same time we assumed[2]:

(2.17) $$\Delta \psi = \frac{\partial \bar{v}'}{\partial x} - \frac{\partial \bar{u}'}{\partial y} = -2 \left( \zeta_{xx} \zeta_{yy} - \zeta_{xy}^2 \right).$$

---

2 Relation (2.17) will be used in the naetre of an equation for the calculation of the function $\psi$.

---

Equation (2.16) determines the form of the arbitrary function $\phi$:

$$\Delta \phi = \frac{\partial}{\partial x} (T_1 \zeta_y) - \frac{\partial}{\partial y} (T_1 \zeta_x).$$

Taking into account, that

$$\frac{\partial}{\partial x} (T_1 \zeta_y) - \frac{\partial}{\partial y} (T_1 \zeta_x) = -\frac{\varepsilon}{m_1^2} \rho \frac{\partial}{\partial \rho} \left( \zeta_{xx} \zeta_{yy} - \zeta_{xy}^2 \right)$$

we obtain

$$\Delta \phi = \frac{\varepsilon}{2 m_1^2} \rho \frac{\partial}{\partial \rho} (\Delta \psi),$$

or with an accuracy to the unessential harmonic function

(2.18) $$\phi = \frac{\varepsilon}{2 m_1^2} \rho \frac{\partial \psi}{\partial \rho}.$$

Therefore the final form of the functions $\overline{u'}$ and $\overline{V'}$ will be

$$\overline{u'} = -\frac{\partial \psi}{\partial y} + 2 \iiint_{(D)} \left[ \left( \frac{\partial \mathfrak{z}}{\partial \eta}, \mathfrak{z}_{x'} \right) - \frac{1}{2} \frac{\partial}{\partial \eta} \left( \frac{\partial \psi}{\partial y'} \right) \right] \eta M_5 \, dD$$

(2.19)
$$\overline{V'} = \frac{\partial \psi}{\partial x} + 2 \iiint_{(D)} \left[ \left( \frac{\partial \mathfrak{z}}{\partial \eta}, \mathfrak{z}_{y'} \right) + \frac{1}{2} \frac{\partial}{\partial \eta} \left( \frac{\partial \psi}{\partial x'} \right) \right] \eta M_5 \, dD$$

Graphs of the influence function $M_5$ are given in figures 1 - 4.

The characteristics of the influence function $M_5$ for $r' \to \infty$ coincide with the characteristics of the function $M_3$ [1].

## III. Calculation of the function $\frac{d\overline{\Omega'}}{dt}$

Now it remains to find the formulas, which are convenient for the calculation of the function $\frac{d\overline{\Omega'}}{dt}$. For this purpose, we shall proceed from the relations

(3.1)
$$\frac{d\overline{\Omega'}}{dt} = \frac{\partial \overline{\Omega'}}{\partial t} + (\mathfrak{z}, \overline{\Omega'}) + O(\varepsilon)$$

Due to the fact, that

$$\overline{\Omega'} = -2(\mathfrak{z}_{xx}\mathfrak{z}_{yy} - \mathfrak{z}_{xy}^2) = -2(\mathfrak{z}_x, \mathfrak{z}_y) = 2(V, u) + O(\varepsilon),$$

(3.2)
$$\frac{d\overline{\Omega'}}{dt} = 2\left[\left(\frac{dV}{dt}, u\right) + \left(V, \frac{du}{dt}\right) + \mathfrak{z}_{xy}(\mathfrak{z}_{xx}\mathfrak{z}_{yy} - \mathfrak{z}_{xy}^2) - \right.$$
$$\left. - \mathfrak{z}_{xy}^2(\mathfrak{z}_{xx} - \mathfrak{z}_{yy})\right] + O(\varepsilon)$$

but

$$\frac{dV}{dt} = -u', \quad \frac{du}{dt} = V'$$

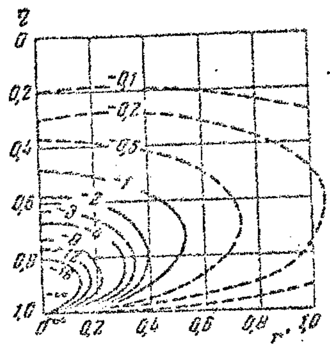

Fig. 1. Influence function $M_5(1,0,\eta,r)$

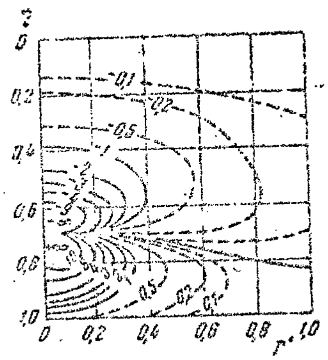

Fig. 2. Influence function $M_5(0,7,\eta,r)$

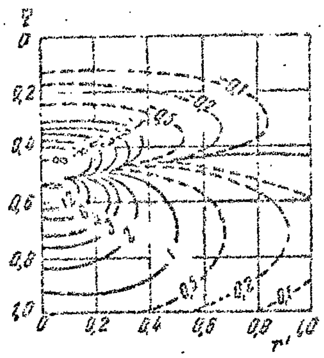

Fig. 3. Influence function $M_5(0,5,\eta,r')$

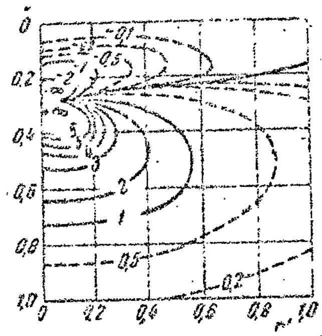

Fig. 4. Influence function $M_5(0,3,\eta,r')$

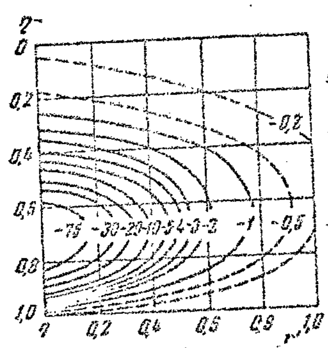

Fig. 5. Influence function $M_6(0,7,\eta,r')$

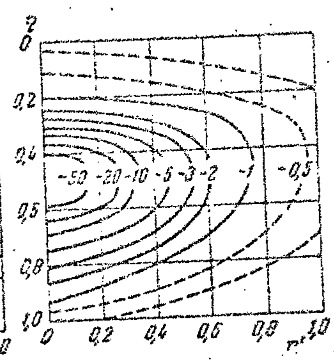

Fig. 6. Influence function $M_6(0,5,\eta,r')$

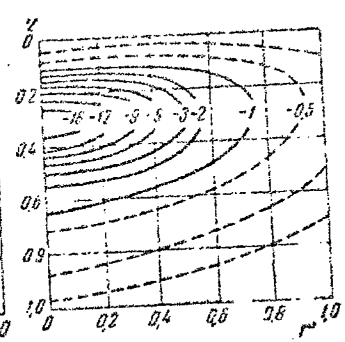

Fig. 7. Influence function $M_6(0,3,\eta,r')$

therefore

$$\frac{d\overline{\Omega'}}{dt} = 2\Big[(u,u')+(v,v')+\zeta_{xy}(\zeta_{xx}\zeta_{yy}-\zeta_{xy}^2)- \tag{3.3}$$
$$-\zeta_{xy}^2(\zeta_{xx}-\zeta_{yy})\Big]+O(\varepsilon)$$

Consequently, taking into account the statement

$$\frac{\partial\overline{\Omega'}}{\partial t}+(\zeta,\overline{\Omega'}) = 2\Big[(\zeta_x,\overline{v'})-(\zeta_y,\overline{u'})+\zeta_{xy}(\zeta_{xx}\zeta_{yy}-\zeta_{xy}^2)-$$
$$-\zeta_{xy}^2(\zeta_{xx}-\zeta_{yy})\Big]+O(\varepsilon)$$

In view of the fact, that in formulas (1.3) and (1.5) it follows that we consider only the principal parts of the expression $\frac{\partial\overline{\Omega}}{\partial t}+(\zeta,\Omega)$, let us consider, that

$$\frac{\partial\overline{\Omega'}}{\partial t}+(\zeta,\overline{\Omega'}) = 2\Big[(\zeta_x,\overline{v'})-(\zeta_y,\overline{u'})+\zeta_{xy}(\zeta_{xx}\zeta_{yy}-\zeta_{xy}^2)- \tag{3.4}$$
$$-\zeta_{xy}^2(\zeta_{xx}-\zeta_{yy})\Big]$$

Therefore, the expression for $A$ and $B$ will now have the form

$$A \doteq -\Big\{2\Big[(\zeta_x,\overline{v'})-(\zeta_y,\overline{u'})+\zeta_{xy}(\zeta_{xx}\zeta_{yy}-\zeta_{xy}^2)$$
$$-\zeta_{xy}^2(\zeta_{xx}-\zeta_{yy})\Big]+\overline{u'}\frac{\partial\Delta\zeta}{\partial x}+\overline{v'}\frac{\partial\Delta\zeta}{\partial y}- \tag{3.5}$$
$$-\Delta\zeta\frac{\partial\tau}{\partial\zeta}+\tau\frac{\partial\Delta\zeta}{\partial\zeta}+\frac{\partial\tau}{\partial x}\frac{\partial^2\zeta}{\partial x\partial\zeta}+\frac{\partial\tau}{\partial y}\frac{\partial^2\zeta}{\partial y\partial\zeta}\Big\},$$
$$B' = -\Big\{\overline{u'}\frac{\partial}{\partial\zeta}\Big(\frac{\partial\zeta}{\partial x}\Big)+\overline{v'}\frac{\partial}{\partial\zeta}\Big(\frac{\partial\zeta}{\partial y}\Big)\Big\}$$

In conclusion, let us perform some transformations of the solution, obtained in [1], for the purpose of bringing them to a more convenient form for practical use. Let us proceed from the formula for the vertical velocities [1], which is non-dimensional quantities has the form

$$\tau = \frac{m_1^2}{\varepsilon}\iiint\limits_{(D)}\Big[\Delta(T,\zeta)+(T,\Delta\zeta)+(\zeta,\Delta T)\Big]\zeta\, M_3\, dD \tag{3.6}$$

Due to the fact, that

$$\Delta(T,\zeta) + (T,\Delta\zeta) + (\zeta,\Delta T) = 2\left[\frac{\partial}{\partial x}(T,\zeta_x) + \frac{\partial}{\partial y}(T,\zeta_y)\right]$$

we have:

$$\tau = \frac{2m_1^2}{\varepsilon}\iiint\limits_{(D)}\left[\frac{\partial}{\partial x}(T,\zeta_x) + \frac{\partial}{\partial y}(T,\zeta_y)\right]\rho M_3\, dD$$

letus integrate by parts and we obtain:

$$\tau = \frac{2m_1^2}{\varepsilon}\iiint\limits_{(D)}(T,\zeta_x)\,\rho\,\frac{\partial M_3}{\partial x}\,dD + \frac{2m_1^2}{\varepsilon}\iiint\limits_{(D)}(T,\zeta_y)\,\rho\,\frac{\partial M_3}{\partial y}\,dD$$

or, taking into account that,

(3.7)
$$\left.\begin{aligned}\frac{\partial M_3}{\partial x} &= \frac{\partial M_3}{\partial r'}\frac{\partial r'}{\partial x}\\ \frac{\partial M_3}{\partial y} &= \frac{\partial M_3}{\partial r'}\frac{\partial r'}{\partial y}\end{aligned}\right\}$$

where

$$\frac{\partial r'}{\partial x} = \frac{x-x'}{r'}, \quad \frac{\partial r'}{\partial y} = \frac{y-y'}{r'}$$

we obtain:

(3.8)
$$\tau = \frac{2m_1^2}{\varepsilon}\iiint\limits_{(D)}\left[(T,\zeta_x)(x-x') + (T,\zeta_y)(y-y')\right]\frac{\rho}{r'}\frac{\partial M_3}{\partial r'}\,dD$$

and

(3.9)
$$\rho M_3 = \frac{1}{2}\sqrt{\frac{\rho}{\eta}}\left\{\delta_1\frac{e^{-\frac{1}{2}\sqrt{\ln^2\frac{\eta}{\rho}+r'^2}}}{\sqrt{\ln^2\frac{\eta}{\rho}+r'^2}} + \delta_2\frac{e^{-\frac{1}{2}\sqrt{\ln^2\frac{\rho}{\eta}+r'^2}}}{\sqrt{\ln^2\frac{\rho}{\eta}+r'^2}} - \frac{e^{-\frac{1}{2}\sqrt{\ln^2\rho\eta+r'^2}}}{\sqrt{\ln^2\rho\eta+r'^2}}\right\}$$

where $\delta_2 = 1-\delta_1$, u kile. $\delta_1 = \begin{cases}1 & \text{at } \rho>\eta\\ 0 & \text{at } \rho<\eta\end{cases}$

Using the symbols of the work [1], expression (3.9) could be written in the following form:

$$\rho M_3 = \frac{1}{2}\sqrt{\frac{\rho}{\eta}}\left\{\delta_1\sigma\left(\frac{\eta}{\rho}, r'\right) + \delta_2\sigma\left(\frac{\rho}{\eta}, r'\right) - \sigma(\rho\eta, r')\right\}. \tag{3.10}$$

Let us form the expression

$$M_6 = \frac{1}{r'}\frac{\partial}{\partial r'}(\rho M_3) = \frac{1}{2}\sqrt{\frac{\rho}{\eta}}\left\{\delta_1\frac{1}{r'}\frac{\partial}{\partial r'}\sigma\left(\frac{\eta}{\rho}, r'\right) + \delta_2\frac{1}{r'}\frac{\partial}{\partial r'}\sigma\left(\frac{\rho}{\eta}, r'\right) - \frac{1}{r'}\frac{\partial}{\partial r'}\sigma(\rho\eta, r')\right\}$$

Due to the fact, that

$$\frac{1}{r'}\frac{\partial}{\partial r'}\sigma(x, r') = \frac{x}{\ln x}\frac{\partial\sigma(x, r')}{\partial x}$$

$$M_6 = \frac{1}{2}\sqrt{\frac{\rho}{\eta}}\left\{\delta_1 a\left(\frac{\eta}{\rho}\right)\sigma_x\left(\frac{\eta}{\rho}, r'\right) + \delta_2 a\left(\frac{\rho}{\eta}\right)\times \sigma_x\left(\frac{\rho}{\eta}, r'\right) - a(\rho\eta)\sigma_x(\rho\eta, r')\right\} \tag{3.11}$$

where

$$a(x) = \frac{x}{\ln x}$$

$$\sigma_x(y, r') = \left.\frac{\partial\sigma(x, r')}{\partial x}\right|_{x=y}$$

Graphs of the influence function $M_6$ are given in figures 5-7.

Thus

$$\tau = \frac{2m_1^2}{E}\iiint_{(D)}\left[(T, \mathfrak{z}_{x'})(x-x') + (T, \mathfrak{z}_{y'})(y-y')\right]M_6\, dD \tag{3.12}$$

It is possible to obtain an analogous formula for $\frac{\partial\tau}{\partial\xi}$ :

$$\frac{\partial\tau}{\partial\xi} = \frac{2m_1^2}{E}\iiint_{(D)}\left[(T, \mathfrak{z}_x)(x-x') + (T, \mathfrak{z}_y)(y-y')\right]\frac{\partial M_6}{\partial\xi}\, dD \tag{3.13}$$

## IV. An example of the calculation of $u'$ and $v'$

The case of 12 May 1951 was considered in the nature of an example of the calculation of $u'$ and $v'$. The synoptic situation of this day was characterized by the presence of an extensive cyclone, situated in the region between Kiev and Smolensk, the cyclonic axis of which was directed almost vertically upward (figures 8-9).

Formulas (2.19) were used for the calculation of $u'$ and $v'$, due to the sufficiently smooth change, through the horizontal, of the integrand function, which depends on the specific distribution of the fields of meteorological elements in comparison with the change of the influence function $\eta \bar{M}_5$, the integration with respectto the horizontal was not carried out, but the following approximate relation was used:

$$u' = -\frac{\partial \psi}{\partial y} + 2\int_0^1 \left[ \left( \frac{\partial \zeta}{\partial \eta}, \zeta_x \right) - \frac{1}{2}\frac{\partial}{\partial \eta}\left( \frac{\partial \psi}{\partial y} \right) \right] \eta \bar{M}_5(\xi, \eta$$

(4.1)

$$v' = \frac{\partial \psi}{\partial x} + 2\int_0^1 \left[ \left( \frac{\partial \zeta}{\partial \eta}, \zeta_y \right) + \frac{1}{2}\frac{\partial}{\partial \eta}\left( \frac{\partial \psi}{\partial x} \right) \right] \eta \bar{M}_5(\xi, \eta$$

where

$$\bar{M}_5(\xi, \eta) = \iint_{(S)} M_5(\xi, \eta, r')\, ds$$

(4.2)

and $S$ is the entire plane $(x, y)$.

For the integration of the equation

$$\Delta \psi = -2\left( \zeta_{xx}\zeta_{yy} - \zeta_{xy}^2 \right)$$

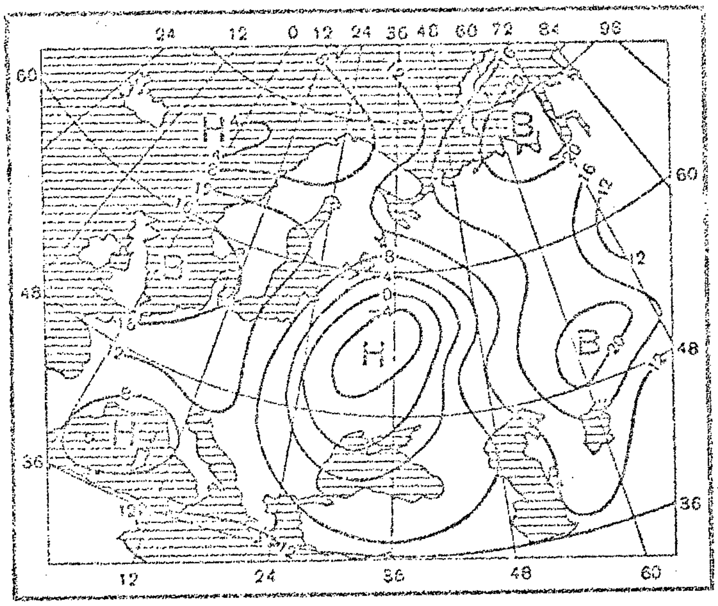

Fig. 8. Map of the baric topography. AT 1000 mb at 12 May 1951.

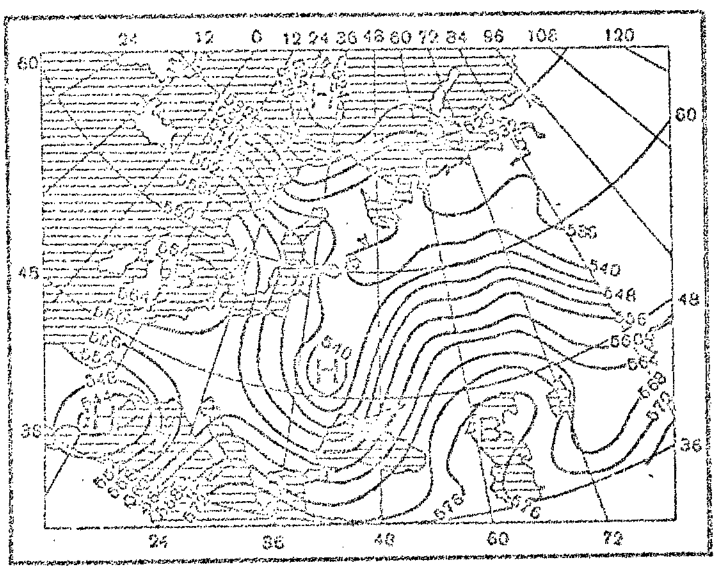

Fig. 9. Map of the baric topography. AT 500 mb at 12 May 1951.

relative to the function $\psi$ on all levels of the atmosphere an approximate method of solution is also used, which is often discussed in the literature. It consists in the fact, that all scales of the spectra of meteorological perturbations can be replaced by some effec tive sinusoidal perturbation of a single frequency for a wave length, more characteristic of the short-range meteorological phenomena (4000 km

In this special case the equations turn out to be quite simple:

$$\psi = 2K^2(z_{xx} z_{yy} - z_{xy}^2) \quad \text{where } K^2 \approx 0.4$$

Calculation of the functions $(z_{xx} z_{yy} - z_{xy}^2)$, $\left(\frac{\partial z}{\partial \eta}, z_x\right.$ $\frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial y}\right)$, etc. was carried out by the finite difference method, while for the horizontal scale unit, the segment of 100 km was taken; the whole interval of change of the functions through the vertical were divided into four parts by the surfaces: 100 850, 700, 500, 300 mb.

The components of the deviations of the wind velocity from the geostrophic wind are present in figures 10-19 (the length scale 1 cm of the velocity arrow — 1 m/sec. ), and the designations are int duced:

$$u'_H = -\frac{\partial \psi}{\partial y}$$

$$v'_H = \frac{\partial \psi}{\partial x}$$

(4.3)

$$u'_g = 2\int_0^1 \left[\left(\frac{\partial z}{\partial \eta}, z_x\right) - \frac{1}{2}\frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial y}\right)\right] \eta\, M_5(\xi, \eta)\, d$$

$$v'_g = 2\int_0^1 \left[\left(\frac{\partial z}{\partial \eta}, z_y\right) + \frac{1}{2}\frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial x}\right)\right] \eta\, M_5(\xi, \eta)\, d$$

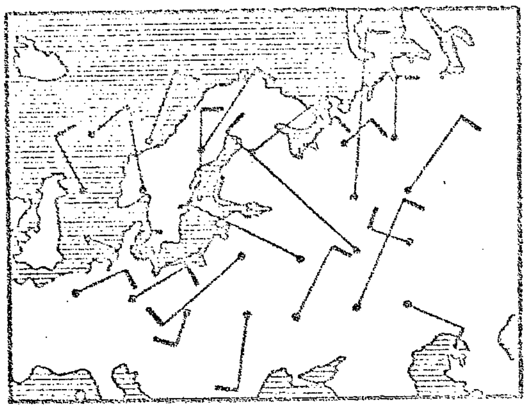

Fig. 10. Components of the deviations of the wind velocity from geostrophic connected with the factor of nonlinearity of the baric fields; AT 1000 mb

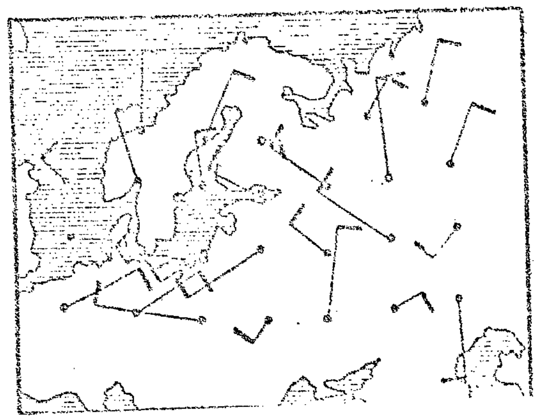

Fig. 11. The same; AT 850 mb

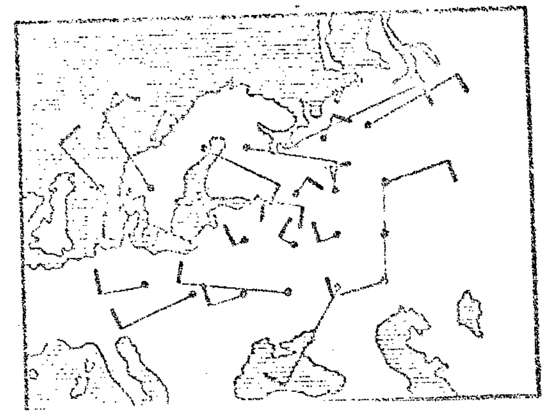

Fig. 12. The same; AT 700 mb

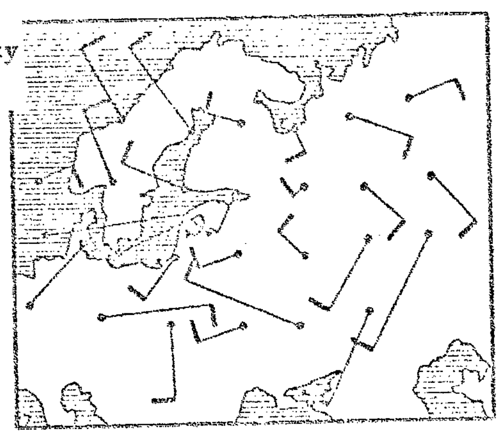

Fig. 13. The same; AT 500 mb

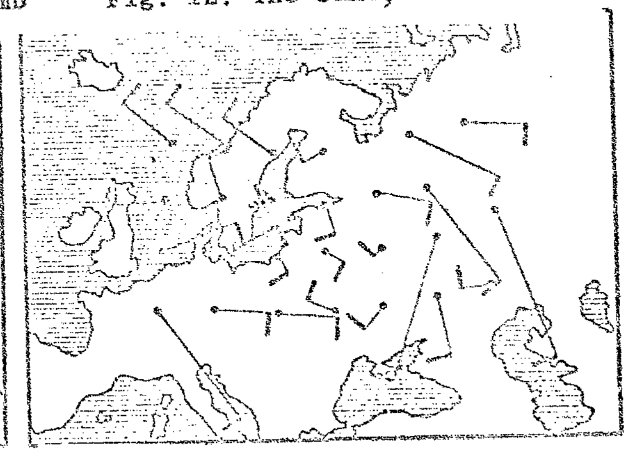

Fig. 14. The same; AT 300 mb

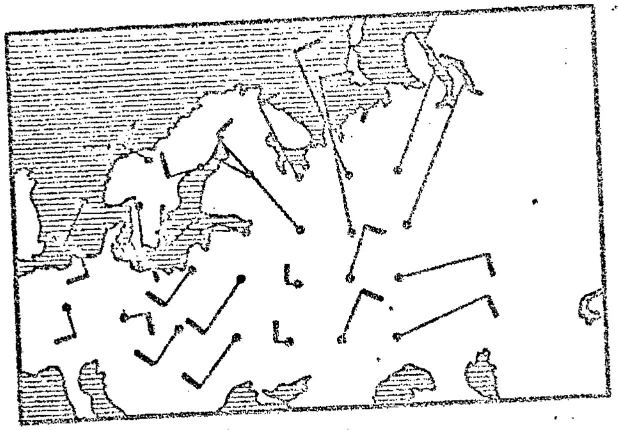

Fig. 15. Components of the deviations of the wind velocity from geostrophic $u_g$ and $v_g$, caused by the presence of the divergence of air parcels in the mean troposphere; AT 1000mb

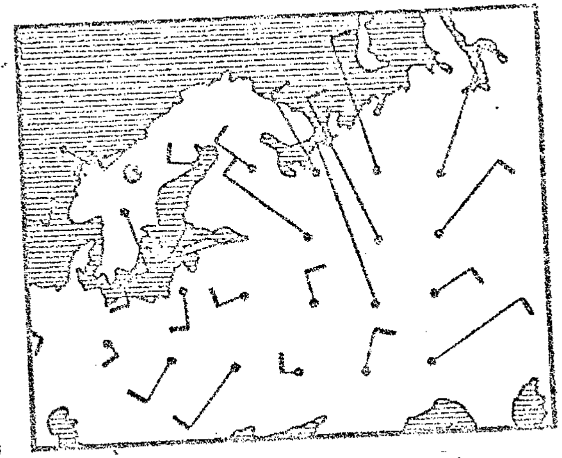

Fig. 16. The same; AT 850 mb

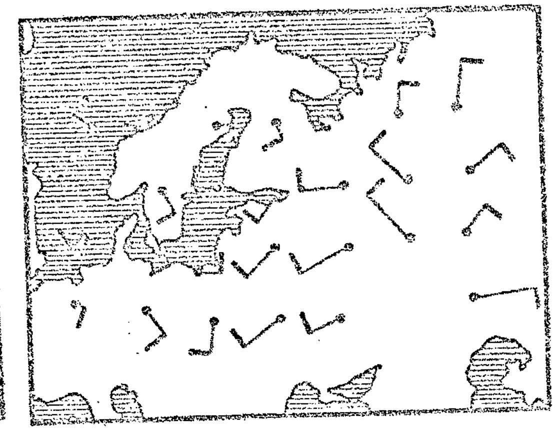

Fig. 17. The same; AT 700 mb

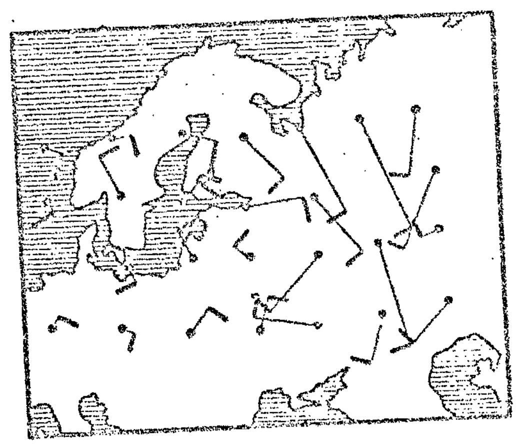

Fig. 18. The same; AT 500 mb

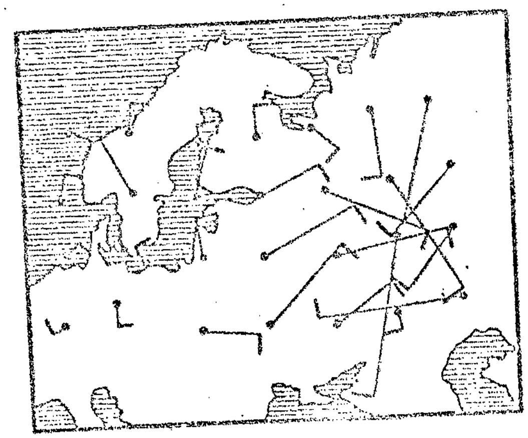

Fig. 19. The same; AT 300 mb

The components $u_H'$ and $v_H'$ are related to the factor of the non-linearity of the baric fields, at the moment when $u_g'$ and $v_g'$, generally speaking, are caused by the presence of the divergence of air particles in the mean troposphere. In order to be convinced of this, it is sufficient, by means of equation (1.2), to form the expressions

$$\Delta u' = -\frac{\partial \Delta \psi}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \tau}{\partial \zeta}\right)$$

$$\Delta v' = \frac{\partial \Delta \psi}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial \tau}{\partial \zeta}\right)$$

The analysis of the example shows, that the components $u_H'$ and $v_H'$ do not change sign with height, and in regions with a cyclonic pressure contour they furnish a negative contribution with respect to the gradient wind, while in regions with an anticyclonic contour - positive.

The functions $u_g'$ and $v_g'$ form a vector, which in the lower layers of the atmosphere is directed, in a cyclonic formation, toward the center, then passes through zero at the mean level $(\zeta \approx 0.$ and in the upper layers of the atmosphere is directed from the center to the periphery of the cyclone. The opposite picture is observed for formations of anticyclonic character.

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